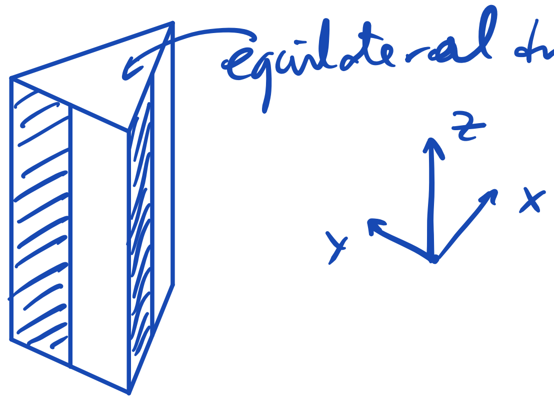
Answermodel resit Symmetry in Physics of June 28, 2022

Exercise 1:



1a) symmetries : - rotations around z axis through center of base over $0^\circ, 120^\circ, 240^\circ$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel \quad \parallel \quad \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad e \quad c \quad c^2$

- reflection in xy plane cutting through middle of rectangular sides
  $\parallel$
  b and all products: bc, bc^2

group of order 6, such that $c^3 = e = b^2$
and $bc = cb \Rightarrow$ Abelian

D_3 is non-Abelian, so cannot be isomorphic

(also $(bc)^2 = e$ for D_3, but $(bc)^2 = c^2 \neq e$ for G_{TB})

1b) Abelian, hence all irreps are 1 dimensional
All elements form conjugacy classes by themselves
So 6 classes = 6 irreps

$\chi(c^3) = \chi(c)^3 = 1$

$\chi(c) = 1, \omega, \omega^2$

$\omega \equiv e^{i2\pi/3}$

	(e)	(c)	(c^2)	(b)	(bc)	(bc^2)
$D^{(1)}$	1	1	1	1	1	1
$D^{(2)}$	1	ω	ω^2	1	ω	ω^2
$D^{(3)}$	1	ω^2	ω	1	ω^2	ω

$\chi(b)^2 = 1$

$\chi(b) = \pm 1$

$D^{(4)}$	1	1	1	-1	-1	-1
$D^{(5)}$	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
$D^{(6)}$	1	ω^2	ω	-1	$-\omega^2$	$-\omega$

$\chi(bc) = \chi(b)\chi(c)$ etc

1D irrep is faithful if all $D^{(i)}(g)$ are distinct

Hence $D^{(5)}$ & $D^{(6)}$ are the only faithful ones

1c) $\chi^V(\theta) = 1 + 2\cos\theta$ for rotations in 3D

$$\Rightarrow \chi^V(e) = 3, \quad \chi^V(c) = 0, \quad \chi^V(c^2) = 0$$

$$D^V(b) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix} \Rightarrow \chi^V(b) = 1$$

$$D^V(bc) = D^V(b)\,D^V(c) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & * & \\ * & \cos\theta & \\ & & 1 \end{pmatrix}$$

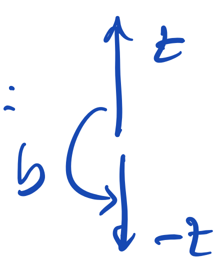
$$= \begin{pmatrix} \cos\theta & * & \\ * & \cos\theta & \\ & & -1 \end{pmatrix} \Rightarrow \chi^V(bc) = 2\cos\theta - 1$$

$\chi^V(bc) = -2$ idem $\chi^V(bc^2) = -2$

$$\Rightarrow \chi^V = (3, 0, 0, 1, -2, -2)$$

$$\langle \chi^{(1)}, \chi^V \rangle = \frac{1}{6}\sum_g \chi^V(g) = \frac{1}{6}\cdot 0 = 0$$

$\Rightarrow$ no invariant vector logical:



$$\chi^A = (3, 0, 0, -1, +2, +2)$$

$\Rightarrow \langle \chi^{(1)}, \chi^{A} \rangle = \frac{1}{6} \cdot 6 = 1$

$\Rightarrow$ invariant axial vector allowed

logical: does opposite to a vector under reflections: $z \uparrow \xrightarrow{b} \uparrow$

2a) invariant tensor: $\sigma' = D^{V} \sigma D^{V\,T} = \sigma$

or equivalently $D^{V} \sigma = \sigma D^{V}$

$\sigma_{ij} = \delta_{ij}$ or $\sigma = \mathbb{1}$ $\quad [D^{V}, \mathbb{1}] = 0$ hence invariant

Or: $\delta_{kl} \to D^{V}_{ik} D^{V}_{jl} \delta_{kl} = D^{V}_{ik} D^{V}_{jk}$

(Exercise Cub of tutorials) $\quad = D^{V}_{ik} D^{V\,T}_{kj} = \delta_{ij}$

$\uparrow$

$D^{V} \in O(3) : D^{V\,T} = (D^{V})^{-1}$

2b) $O \in O(3) \Rightarrow \det(O) = \pm 1$

$\det(O) = +1$ are rotations

$\det(O) = -1$ are reflections $\leftarrow P = -\mathbb{1}_{3\times 3}$

$\det(P) = -1$

$\forall O \in O(3)$ with $\det(O) = -1$, $\det(PO) = +1$.

Call $PO = R \Rightarrow O = PR$ $\qquad \det +1 \underset{P}{\overset{P}{\rightleftarrows}} \det -1$

Hence all reflections O can be written as $-\mathbb{1}_{3\times 3}$ times a rotation R.

For example, reflection in a plane in $\mathbb{R}^3$

take x,y plane for definiteness:

$$O = \begin{pmatrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & -1 \end{pmatrix} = \begin{pmatrix} -1 & & \emptyset \\ & -1 & \\ \emptyset & & -1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & & \emptyset \\ & -1 & \\ \emptyset & & 1 \end{pmatrix}}_{\left.\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|_{\theta=\pi}}$$

$= -\mathbb{1} \times$ rotation around z axis over π

2c) $\underset{\substack{\uparrow \\ \text{vector}}}{\vec{u}} \cdot \underbrace{(\vec{v} \times \vec{w})}_{\text{axial vector}}$ is oriented volume spanned by 3 3-vectors

under rotations it is like $\vec{u} \cdot \vec{v}$: scalar

under reflection $P = -\mathbb{1}$: $\vec{u} \to -\vec{u}$ etc

$\vec{u} \cdot (\vec{v} \times \vec{w}) \to (-1)^3 \vec{u} \cdot (\vec{v} \times \vec{w}) = -\vec{u} \cdot (\vec{v} \times \vec{w})$

so opposite to $\vec{u} \cdot \vec{v}$: pseudoscalar

3a) $E(2)$ = Euclidean group = group of isometries in $\mathbb{R}^2$

(see chapter 1.6 of lecture notes)

isometry: distance preserving transformation

In $\mathbb{R}^2$: rotations, reflections (in a line!), translations & glide reflections

3b) elements of $E(2)$: $(O|\vec{a})$ with $O \in O(2)$, $\vec{a} \in \mathbb{R}^2$

such that $(O|\vec{a})\vec{x} = O\vec{x} + \vec{a}$

$\Rightarrow (O_1|\vec{a}_1)(O_2|\vec{a}_2)\vec{x} = (O_1|\vec{a}_1)(O_2\vec{x} + a_2)$

$= O_1(O_2\vec{x} + \vec{a}_2) + a_1 = O_1O_2\vec{x} + O_1\vec{a}_2 + \vec{a}_1$

$= (O_1O_2 \mid O_1\vec{a}_2 + \vec{a}_1)$

3c) embed $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

write $(O \mid \vec{a})$ as $\begin{pmatrix} O_{11} & O_{12} & a_1 \\ O_{21} & O_{22} & a_2 \\ 0 & 0 & 1 \end{pmatrix}$ 3D rep

(see exercise 3.6 of Jones)

no 2D rep is possible since it would mean $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ which is a linear transformation, whereas translations are not linear